SOLUTIONS OF EXERCISES.

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About the vertices of an equilateral triangle three spheres are drawn with radii equal to the side of the triangle. Find the volume common to them all.

[*W. M. Thornton.*]

SOLUTION.

Let ABC be the triangle of the centres, O its ortho-centre, and N one of the apices of the solid. Draw BO to meet the surface in Q. The planes NOA, NOQ, AOQ, and the sphere-surface ANQ bound one-twelfth of the solid. The area of the sphere-surface ANQ is the difference between the area of the zone-surface ACN, whose angle ACN is arc tan $2\sqrt{2}$, and that of the right spherical triangle QCN, whose angles are arc tan $2\sqrt{2}$ and arc cot $2\sqrt{2}$.

The excess of this triangle is

$$E = \text{arc tan } 2\sqrt{2} + \text{arc cot } \sqrt{2} - \tfrac{1}{2}\pi.$$

Its area is

$$\Delta = ER^2,$$

R being the radius of the sphere. The area of the zone-piece is

$$Z = \tfrac{1}{2}R^2 \text{ arc tan } 2\sqrt{2}.$$

Hence the curved surface of the portion of the solid under consideration is

$$\begin{aligned} Z - \Delta &= R^2\left(\tfrac{1}{2}\pi - \text{arc cot } \sqrt{2} - \tfrac{1}{2}\text{ arc tan } 2\sqrt{2}\right) \\ &= R^2\left(\text{arc tan } \sqrt{2} - \tfrac{1}{2}\text{ arc tan } 2\sqrt{2}\right). \end{aligned}$$

The volume of the spherical pyramid which has this surface for its base is

$$\tfrac{1}{3}R^3\left(\text{arc tan } \sqrt{2} - \tfrac{1}{2}\text{ arc tan } 2\sqrt{2}\right).$$

The volume of the cone-segment whose base is NOA and apex B is

$$\tfrac{1}{3}R^3\left(\tfrac{1}{4}\text{ arc tan } 2\sqrt{2} - \tfrac{1}{24}\sqrt{2}\right).$$

The difference between these volumes is one-twelfth the volume of the solid common to three spheres. Hence

$$V = R^3\left(\tfrac{1}{12}\sqrt{2} + 4\text{ arc tan } \sqrt{2} - 3\text{ arc tan } 2\sqrt{2}\right).$$

[*W. H. Echols.*]

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If

$$J = \begin{vmatrix} a_1 & b_1 & c_1 & \dots & h_1 \\ a_2 & b_2 & c_2 & \dots & h_2 \\ a_3 & b_3 & c_3 & \dots & h_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & h_n \end{vmatrix}, \text{ and } D = \left\| \begin{matrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_2 & h_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_3 & h_3 \end{vmatrix} \\ \dots & \dots & \dots & \dots \\ \begin{vmatrix} a_1 & b_1 \\ a_n & b_n \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_n & c_n \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_n & h_n \end{vmatrix} \end{matrix} \right\| ;$$

then will $$D = a_1^{n-2} J.$$ [*T. M. Blakslee.*]

SOLUTION.

This exercise is given on p. 77 of Muir's Theory of Determinants. It may be got from the result of section 53 of that work; or it may be derived by a process analogous to that of the section referred to, as follows: Multiply each column after the first by a_1; add to each element of the second column, thus multiplied, $-b_1$ times the corresponding element of the first column; to each element of the new third column $-c_1$ times the corresponding element of the first column; ... to each element of the new nth column $-h_1$ times the corresponding element of the first column; and we have

$$a_1^{n-1} J = \begin{vmatrix} a_1 & 0 & 0 & \dots & 0 \\ a_2 & -a_2 b_1 + a_1 b_2 & -a_2 c_1 + a_1 c_2 & \dots & -a_2 h_1 + a_1 h_2 \\ a_3 & -a_3 b_1 + a_1 b_3 & -a_3 c_1 + a_1 c_3 & \dots & -a_3 h_1 + a_1 h_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & -a_n b_1 + a_1 b_n & -a_n c_1 + a_1 c_n & \dots & -a_n h_1 + a_1 h_n \end{vmatrix}$$

$$= a_1 \left\| \begin{matrix} \begin{vmatrix} a_1 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_2 \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_3 \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_3 \end{vmatrix} \\ \dots & \dots & \dots & \dots \\ \begin{vmatrix} a_1 & b_n \end{vmatrix} & \begin{vmatrix} a_1 & c_n \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_n \end{vmatrix} \end{matrix} \right\| = a_1 D;$$

$$\therefore a_1^{n-2} J = D.$$

[*W. B. Richards.*]

Also solved by L. G. Weld and the proposer.

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Show that the attraction of a finite mass on one of its points is finite.

[*A. Hall.*]

SOLUTION.

Take the origin at the point attracted, and let $k\,dx\,dy\,dz$ be the element of mass, V the potential for a particle of the body, and r the distance of the particle from the origin. We have

$$V=\int\int\int\frac{k\,dx\,dy\,dz}{r}.$$

Put

$$x=r\cos u,\qquad y=r\sin u\cos\lambda,\qquad z=r\sin u\sin\lambda;$$

then

$$dx\,dy\,dz=r^2\sin u\,du\,d\lambda\,dr,$$

and

$$V=\int\int\int kr\sin u\,du\,d\lambda\,dr.$$

The limits of integration are $u=0$, to $u=\pi$; $\lambda=0$, to $\lambda=2\pi$; and $r=0$, to the limits of the body. For the component of the force in the axis of x we have

$$X=\frac{\partial V}{\partial x}=\int\int\int\frac{kx\,dx\,dy\,dz}{r^3}=\int\int\int k\cos u\sin u\,du\,d\lambda\,dr,$$

with similar values for Y and Z. These components are finite, and therefore the resultant is finite. See Gauss, *Allgemeine Lehrsätze, etc.*

[*A. Hall.*]

EXERCISES.

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A body at distance r from the sun is moving with velocity v. Prove that the major axis of the orbit described is parallel to the direction of motion if, and only if, the velocity is "circular velocity for the distance r."

[*Ellery W. Davis.*]

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If a horizontal beam of length $2a$ is supported at each end, and has a load in the form of an isosceles triangle, base $2a$, height b, a unit's thickness throughout, and heaviness unity; show that the deflection of the beam due to this triangular load is $\dfrac{2a^4b}{15EI}$.

[*T. U. Taylor.*]